VFAST Transactions on Mathematics
http://www.vfast.org/index.php/VTM@ISSN: 2309-0022
Volume 1, Number 1, May-June 2013

pp. **21–25**

SOME PROPERTIES OF MEROMORPHIC ALPHA-CONVEX FUNCTIONS AND ITS APPLICATIONS

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(Revised June 5, 2013)

Abstract. *The aim of the present paper is to obtain sufficient condition for the class of meromorphic alpha convex functions of order* ξ and then to study mapping properties of an integral operator. Many known results apear as special consequences of our work.

Keywords: Meromorphic alpha convex functions; Inteegral operator

1. **Introduction.** Let $\Sigma(n)$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k\, z^k, \tag{1.1}$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\}$. In particular, $\Sigma(1) = \Sigma$. For λ is real with $|\lambda| < \frac{\pi}{2}$, $\alpha \geq 0$, $0 \leq \xi < 1$, $n \in \mathbb{N}$, we denote by $\Sigma\mathcal{S}(\lambda, n, \xi)$, $\Sigma\mathcal{C}(\lambda, n, \xi)$ and $\Sigma\mathcal{M}(\lambda, n, \alpha, \xi)$, the suclasses of $\Sigma(n)$ consisting of all meromorphic functions of the form (1.1) which are defined, respectively, by

$$-Re e^{i\lambda} \frac{zf'(z)}{f(z)} > \xi \cos\lambda,\ (z \in \mathbb{U}^*), \tag{1.2}$$

$$-Re e^{i\lambda} \frac{(zf'(z))'}{f'(z)} > \xi \cos\lambda,\ (z \in \mathbb{U}^*), \tag{1.3}$$

$$-Re e^{i\lambda} \left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)}\right\} > \xi \cos\lambda,\ (z \in \mathbb{U}^*). \tag{1.4}$$

Making $\lambda = 0$, $n = 1$ in (1.2), (1.3) and (1.4), we get the well-known subclasses of Σ consisting of meromorphic functions which are starlike, convex and alpha convex of order ξ $(0 \leq \xi < 1)$ respectively. For detail of the classes defined by (1.2), (1.3), (1.4) and related topics, we refer the work of Rosihan and Ravichandran [1], Goyal and Prajapat [2], Joshi and Srivastava [3], Liu and Srivastava [4], Raina and Srivastava [5], Xu and Yang [6] and Owa et al [7].

For $f(z) \in \Sigma$, Wang [8] and Nehari and Netanyahu [9] introduced and studied the subclass $\Sigma_N(\tau)$ of Σ consisting of functions $f(z)$ satisfying

$$-Re \frac{(zf'(z))'}{f'(z)} < \tau,\ (\tau > 1,\ z \in \mathbb{U}^*).$$

We now define a subclass $\Sigma\mathcal{N}(\lambda, n, \alpha, \tau)$ of $\Sigma(n)$ consisting of functions $f(z)$ of the form (1.1) satisfying

$$-Ree^{i\lambda}\left((1-\alpha)\frac{zf'(z)}{f'(z)}+\alpha\frac{(zf'(z))'}{f'(z)}\right)<\tau\cos\lambda,\ (\tau>1,\ z\in\mathbb{U}^*). \tag{1.5}$$

Integral operators for different classes of analytic, univalent in the open unit disk are studied by various authors, see [10, 11, 12, 13, 14, 15, 16]. We now consider the following general integral operator of meromorphic functions

$$G_m(z)=I_m(\delta,\alpha_j;f_j(z))=\left\{\frac{\delta}{z^2}\int_0^z t^{\delta-1}\prod_{j=1}^m(t(f_j(t)))^{\alpha_j}\,dt\right\}^{\frac{1}{\delta}}. \tag{1.6}$$

For $\delta=1$, we obtain the integral operator $I_m(f_j(z))$ introduced and studied by Mohammed and Darus [17].

Sufficient condition were studied by various authors for different subclasses of analytic and multivalent functions, for some of the related work see [18, 19, 20, 21]. The object of the present paper is to obtain sufficient conditions for the class $\Sigma\mathcal{M}(\lambda, n, \alpha, \xi)$ and then study mapping properties of the integral operator given by (1.6).

We will assume throughout our discussion, unless otherwise stated, that λ is real with $|\lambda|<\frac{\pi}{2}$, $0\le\xi<1$, $\tau>1$, $n\in\mathbb{N}$, $\alpha_j>0$ for $j\in\{1,\dots m\}$, $\delta>0$, $\alpha\ge 0$ and

$$J_\alpha(f)=(1-\alpha)\frac{zf'(z)}{f'(z)}+\alpha\frac{(zf'(z))'}{f'(z)}. \tag{1.7}$$

To obtain our main results, we need the following Lemma.

Lemma 1.1 [21]**.** If $q(z)\in\Sigma(n)$ with $n\ge 1$ and satisfies the condition

$$\left|z^2q'(z)+1\right|<\frac{n}{\sqrt{n^2+1}}\quad(z\in\mathbb{U}^*),$$

then

$$q(z)\in\Sigma\mathcal{S}(n).$$

2. Sufficiency criteria for the class. $\Sigma\mathcal{M}(\lambda, n, \alpha, \xi)$

Theorem 2.1. If $f(z)\in\Sigma(n)$ satisfies

$$\left|\left\{\left(zf(z)\left(\frac{-zf'(z)}{f(z)}\right)^\alpha\right)^{\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}}e^{i\lambda}J_\alpha(f)+\xi\cos\lambda+i\sin\lambda\right\}\right.$$
$$\left.+(1-\xi)\cos\lambda\right|<\frac{n}{\sqrt{n^2+1}}(1-\xi)\cos\lambda\quad(z\in\mathbb{U}^*), \tag{2.1}$$

then $f(z)\in\Sigma\mathcal{M}(\lambda, n, \alpha, \xi)$, where $J_\alpha(f)$ is given by (1.7).

Proof. Let us set a function $q(z)$ by

$$q(z)=\frac{1}{z}\left(zf(z)\left(\frac{-zf'(z)}{f(z)}\right)^\alpha\right)^{\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}}=\frac{1}{z}+\frac{\alpha\ e^{i\lambda}\ a_n\ b_n}{(1-\xi)\cos\lambda}z^n+\dots \tag{2.2}$$

for $f(z)\in\Sigma(n)$. Then clearly (2.2) shows that $q(z)\in\Sigma(n)$.

Logarithmic differentiating of (2.2) gives

$$\frac{q'(z)}{q(z)}=\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}\left[(1-\alpha)\frac{f'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{zf'(z)}+\frac{1}{z}\right]-\frac{1}{z} \tag{2.3}$$

which further implies

$$\left|z^2q'(z)+1\right|=\left|\left(zf(z)\left(\frac{-zf'(z)}{f(z)}\right)^\alpha\right)^{\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}}\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}\right.$$
$$\left.[J_\alpha(f)+\xi\cos\lambda+i\sin\lambda]+1\right|.$$

Thus using (2.1), we get

$$\left|z^2q'(z)+1\right| \leq \frac{n}{\sqrt{n^2+1}}, \ \ (z\in\mathbb{U}^*).$$

Therefore by Lemma 1.1, we have $q(z)\in\Sigma\mathcal{S}\left(n\right)$.

From (2.3), we can write

$$\frac{zq'(z)}{q(z)}=\frac{1}{(1-\xi)\cos\lambda}\left[e^{i\lambda}J_{\alpha}\left(f\right)+\xi\cos\lambda+i\sin\lambda\right].$$

Since $q(z)\in\Sigma\mathcal{S}\left(n\right)$, it implies that $Re\left(-\frac{zq'(z)}{q(z)}\right)>0$. Therefore, we get

$$\frac{1}{(1-\xi)\cos\lambda}\left[-Ree^{i\lambda}J_{\alpha}\left(f\right)-\xi\cos\lambda\right]=Re\left(-\frac{zq'(z)}{q(z)}\right)>0$$

or

$$-Ree^{i\lambda}J_{\alpha}\left(f\right)>\xi\cos\lambda.$$

and therefore $f(z)\in\Sigma\mathcal{M}\left(\lambda,n,\alpha,\xi\right)$.

By taking $\alpha=0$ and $\alpha=1$ in Theorem 2.1, we obtain Corollary 2.2 and Corollary 2.3 respectively.

Corollary 2.2. If $f(z)\in\Sigma\left(n\right)$ satisfies

$$\left|\left(zf(z)\right)^{\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}+\xi\cos\lambda+i\sin\lambda\right\}+(1-\xi)\cos\lambda\right|$$

$$<\frac{n(1-\xi)\cos\lambda}{\sqrt{(n^2+1}}\ (z\in\mathbb{U}^*), \tag{2.4}$$

then $f(z)\in\Sigma\mathcal{S}\left(\lambda,n,\xi\right)$.

Corollary 2.3. If $f(z)\in\Sigma\left(n\right)$ satisfies

$$\left|\left(-z^2f'(z)\right)^{\frac{e^{i\lambda}}{(1-\xi)\cos\lambda}}\left\{e^{i\lambda}\left(\frac{zf''(z)}{f'(z)}+1\right)+\xi\cos\lambda+i\sin\lambda\right\}\right.$$

$$\left.+(1-\xi)\cos\lambda\right|\frac{n}{\sqrt{(n^2+1}}(1-\xi)\cos\lambda,\quad \left(z\in\mathbb{U}^*\right), \tag{2.5}$$

then $f(z)\in\Sigma\mathcal{C}\left(\lambda,n,\xi\right)$.

3. Mapping properties of the integral operator. $G_m\left(z\right)$.

Theorem 3.1. For $j\in\{1,\dots m\}$, let $f_j\left(z\right)\in\Sigma\left(n\right)$ and satisfy (2.4). If

$$\,_{j=1}^{m}\alpha_j<\frac{2\left(2-\delta\right)}{\left(1-\xi\right)},\ 0<\delta<2, \tag{3.1}$$

then $G_m\left(z\right)\in\Sigma\mathcal{N}\left(\lambda,n,\delta,\eta\right)$ with $\eta>1$ and $G_m\left(z\right)$ is given by (1.6).

Proof. From (1.6), we obtain

$$\delta z^2G_m^{\delta-1}\left(z\right)G_m'\left(z\right)+2zG_m^{\delta}\left(z\right)=\delta z_{j=1}^{\delta-1m}\left(zf_j\left(z\right)\right)^{\alpha_j}.$$

Divide both sides by $zG_m^{\delta-1}\left(z\right)$, we have

$$\delta zG_m'\left(z\right)+\left(p+1\right)G_m\left(z\right)=\delta z^{\delta-2}G_m^{1-\delta}\left(z\right)_{j=1}^{m}\left(zf_j\left(z\right)\right)^{\alpha_j}.$$

Differentiating again logarithmically, we have

$$\frac{\delta zG_m''\left(z\right)+\left(\delta+2\right)G_m'\left(z\right)}{\delta zG_m'\left(z\right)+2G_m\left(z\right)}=\left(\delta-2\right)\frac{1}{z}+\left(1-\delta\right)\frac{G_m'\left(z\right)}{G_m\left(z\right)}+_{j=1}^{m}\alpha_j\left(\frac{f_j'\left(z\right)}{f_j\left(z\right)}+\frac{1}{z}\right). \tag{3.2}$$

Now by simple computation, we get

$$\left(1-\frac{1}{\delta}\right)\frac{zG_m'\left(z\right)}{G_m\left(z\right)}+\frac{1}{\delta}\frac{\left(zG_m'\left(z\right)\right)'}{G_m'\left(z\right)}=\frac{1}{\delta}{}_{j=1}^{m}\alpha_j\left(\frac{zf_j'\left(z\right)}{f_j\left(z\right)}+1\right)-\frac{1}{\delta}\left(4-\delta\right)$$

$$+\frac{G_m\left(z\right)}{zG_m'\left(z\right)}\left[2_{j=1}^{m}\alpha_j\left(\frac{zf_j'\left(z\right)}{f_j\left(z\right)}+1\right)+2\left(\delta-2\right)\right],$$

or, equivalently we have

$$-e^{i\lambda}\left\{\left(1-\frac{1}{\delta}\right)\frac{zG_m'(z)}{G_m(z)}+\frac{1}{\delta}\frac{(zG_m'(z))'}{G_m'(z)}\right\}=\frac{1}{\delta}\sum_{j=1}^{m}\alpha_j\left(-e^{i\lambda}\frac{zf_j'(z)}{f_j(z)}-e^{i\lambda}\right)$$
$$+\frac{1}{\delta}(4-\delta)e^{i\lambda}+\frac{G_m(z)}{zG_m'(z)}\left[2-\delta-\sum_{j=1}^{m}\alpha_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right)\right](1+1)e^{i\lambda},$$

By taking real part on both sides, we obtain

$$-Ree^{i\lambda}\left\{\left(1-\frac{1}{\delta}\right)\frac{zG_m'(z)}{G_m(z)}+\frac{1}{\delta}\frac{(zG_m'(z))'}{G_m'(z)}\right\}=\frac{1}{\delta}\sum_{j=1}^{m}\alpha_j\left(-Ree^{i\lambda}\frac{zf_j'(z)}{f_j(z)}-\cos\lambda\right)$$
$$+\frac{1}{\delta}(4-\delta)\cos\lambda+Re\frac{G_m(z)}{zG_m'(z)}\left[(2-\delta)-\sum_{j=1}^{m}\alpha_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right)\right](1+1)e^{i\lambda},$$

which further implies that

$$-Ree^{i\lambda}\left\{\left(1-\frac{1}{\delta}\right)\frac{zG_m'(z)}{G_m(z)}+\frac{1}{\delta}\frac{(zG_m'(z))'}{G_m'(z)}\right\}\leq\frac{1}{\delta}\sum_{j=1}^{m}\alpha_j\left(-Ree^{i\lambda}\frac{zf_j'(z)}{f_j(z)}-\cos\lambda\right)$$
$$+\frac{1}{\delta}(4-\delta)\cos\lambda+\left|\frac{G_m(z)}{zG_m'(z)}\left[(2-\delta)-\sum_{j=1}^{m}\alpha_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right)\right]2e^{i\lambda}\right|.$$

Let

$$\eta=\left|\frac{G_m(z)}{zG_m'(z)}\left[(2-\delta)-\sum_{j=1}^{m}\alpha_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right)\right]2e^{i\lambda}\right|$$
$$+\frac{1}{\delta}(4-\delta)+\frac{1}{\delta}\sum_{j=1}^{m}\alpha_j\left(-\frac{1}{\cos\lambda}Ree^{i\lambda}\frac{zf_j'(z)}{f_j(z)}-1\right).$$

Clearly we have

$$\eta>\frac{1}{\delta}(4-\delta)+\frac{1}{\delta\cos\lambda}\sum_{j=1}^{m}\alpha_j\left(-Ree^{i\lambda}\frac{zf_j'(z)}{f_j(z)}-\cos\lambda\right).$$

Then by using (3.1) and Theorem 2.1 with $\alpha=0$, we obtain

$$\eta>\frac{1}{\delta}\left[\sum_{j=1}^{m}\alpha_j(\xi-1)+(4-\delta)\right]>1.$$

Therefore $G_m(z)\in\Sigma\mathcal{N}(\lambda,n,\delta,\eta)$ with $\eta>1$.

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